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DETERMINACY IMPLIES THAT $\aleph_2$ IS SUPERCOMPACT

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ABSTRACT
Assuming the axiom of determinacy, $\aleph_1$ and $\aleph_2$ are δ-supercompact, where δ is a fairly large cardinal which will be defined below.

We work in ZF plus the axiom of dependent choice. We do not assume the full axiom of choice.

DEFINITION. For κ a cardinal and λ an ordinal, $P_\kappa(\lambda)$ is the set of all subsets of λ of cardinality less than κ. A filter $\mathscr{F}$ on $P_\kappa(\lambda)$ is *fine* if for all η less than λ, $\{S \in P_\kappa(\lambda) : \eta \in S\} \in \mathscr{F}$. A filter $\mathscr{F}$ on $P_\kappa(\lambda)$ is *normal* if whenever $G : P_\kappa(\lambda) \to \lambda$ is a function such that $\{S : G(S) \in S\} \in \mathscr{F}$ then there is an η such that $\{S : G(S) = \eta\} \in \mathscr{F}$, i.e., every choice function on $P_\kappa(\lambda)$ is constant almost everywhere. A cardinal κ is λ*-supercompact* if there is an ultrafilter $\mathscr{U}$ on $P_\kappa(\lambda)$ such that

(1) $\mathscr{U}$ is κ-complete,

(2) $\mathscr{U}$ is fine,

(3) $\mathscr{U}$ is normal.

If $\lambda < \lambda'$ and κ is λ'-supercompact then it is λ-supercompact, and κ is κ-supercompact iff κ is measurable.

Supercompactness has been studied in two different contexts: assuming the axiom of choice and assuming the axiom of determinacy (AD). The two subjects do not seem to have much in common. Choice implies, among other things, that supercompact cardinals are very large; under AD this is not the case. For information on supercompact cardinals, assuming the axiom of choice, see [5], [6], or [15].

In this paper we will be concerned solely with the choiceless world of AD. For information on this axiom, see [12] or [5]; [12] is also the basic reference for all unexplained concepts from descriptive set theory that appear here.

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Let Γ be the pointclass $(\Sigma_1^2)^{L[\mathbf{R}]}$. Then $\boldsymbol{\Gamma}, \Delta, \boldsymbol{\Delta}$, etc. have the usual meaning, following [12].

Let $\boldsymbol{\delta} = (\boldsymbol{\delta}_1^2)^{L[\mathbf{R}]} = \sup\{$order type of $\lesssim$: $\lesssim$ is a prewellordering of $\mathbf{R}$ and $\lesssim$ is in $\boldsymbol{\Delta}\}$.

Assuming AD, $\boldsymbol{\delta}$ is rather large. For example, it is a weakly Mahlo cardinal — but not the first weakly Mahlo cardinal ([8], theorem 3.1).

The pointclass Γ has the uniformization property (defined in [12], page 235), assuming AD. Assuming AD + $V = L[\mathbf{R}]$, there is a relation in $\check{\Gamma} = \Pi_1^2$ that cannot be uniformized by any function; hence $\boldsymbol{\Gamma}$ is the largest class for which uniformization can be proved from AD alone. This is the reason for the importance of Γ and $\boldsymbol{\delta}$ in descriptive set theory. For details, see [10].

THEOREM 1. (AD; Solovay, Harrington–Kechris [4]). *$\aleph_1$ is $\boldsymbol{\delta}$-supercompact.*

Theorem 1 is not new. It was first proved by Solovay from $AD_{\mathbf{R}}$. His proof involved a game on reals. Harrington and Kechris proved it from AD, by proving in AD alone that this particular type of game on reals is determined, and then following the Solovay proof. We give a new proof of Theorem 1 here, a proof that does not use Solovay's game. This new proof, unlike the original one, can be modified to yield the following theorem, which we also prove here.

THEOREM 2. (AD). *$\aleph_2$ is $\boldsymbol{\delta}$-supercompact.*

Other work on supercompactness and its applications, assuming AD, can be found in the following references: [1], [2], [3], [4] [7], [14], [16].

The proofs of Theorems 1 and 2 follow. For the rest of the paper, *AD is always assumed.*

For the benefit of the reader unfamiliar with the pointclass Γ, but familiar with the analytical and projective pointclasses, the following remark may be helpful. The pointclass Γ has a structure theory very much like that of Σ_{2n}^1, a theory which is explained in great detail in [12]. All structure properties relevant to the proof of Theorems 1 and 2 are the same. If we let $\Gamma = \Sigma_{2n}^1$ and $\boldsymbol{\delta} = \boldsymbol{\delta}_{2n-1}^1$, rather than letting $\Gamma = (\Sigma_1^2)^{L[\mathbf{R}]}$ and $\boldsymbol{\delta} = (\boldsymbol{\delta}_1^2)^{L[\mathbf{R}]}$, then what follows is a proof that $\aleph_1$ and $\aleph_2$ are $\boldsymbol{\delta}_{2n-1}^1$-supercompact.

Code ordinals less than $\boldsymbol{\delta}$ by a fixed Γ-norm on a complete Γ-set. The ordinal encoded by the real z is denoted $|z|$. The binary relation on reals, $x \in \Delta(y)$, is transitive, and hence gives rise to Δ-degrees. There is a countably complete ultrafilter $\mathcal{V}$ on the set of Δ-degrees, namely the filter generated by the cones (the Martin measure — [9], [12], 7D.17). If $\boldsymbol{d}$ is a Δ-degree, then let

$$A_{\boldsymbol{d}} = \{\xi < \boldsymbol{\delta} : (\exists z \in \mathbf{R})(|z| = \xi \;\&\; z \text{ is } \Delta\text{-in-}\boldsymbol{d})\}.$$

Let $\mathcal{U}_1$ be the ultrafilter on $P_{\aleph_1}(\boldsymbol{\delta})$ obtained from the map $\boldsymbol{d} \mapsto A_{\boldsymbol{d}}$ and the ultrafilter $\mathcal{V}$.

PROPOSITION 1.1. *$\mathcal{U}_1$ is a countably complete fine ultrafilter on $P_{\aleph_1}(\boldsymbol{\delta})$.*

LEMMA 1.2. *$\mathcal{U}_1$ is normal.*

PROOF. Suppose not. Let $G : P_{\aleph_1}(\boldsymbol{\delta}) \to \boldsymbol{\delta}$ be a choice function that is not constant $\mathcal{U}_1$-a.e. Let $F : \boldsymbol{\delta} \to \mathrm{Pow}(\mathbf{R})$ be the following function:

$$\forall \xi < \boldsymbol{\delta},\ x \in F(\xi) \text{ iff}$$
$$(\text{for all } \Delta\text{-degrees } \boldsymbol{d})(\text{if } x \text{ is } \Delta\text{-in-}\boldsymbol{d} \text{ then } G(A_{\boldsymbol{d}}) \neq \xi).$$

Since G is not constant a.e., for all ξ, $F(\xi) \neq \varnothing$. By the Moschovakis Coding Lemma ([11], [12] theorem 7D.5), there is a choice subfunction F' of F such that F' is $\boldsymbol{\Gamma}$-in-the-codes. Let y be a real such that F' is $\Gamma(y)$ and let $\boldsymbol{d}$ be the Δ-degree of y. Let $\xi = G(A_{\boldsymbol{d}})$. Then there is a $z \in \Delta(y)$ such that $|z| = \xi$. Since F' is a $\Gamma(y)$ relation in $\mathbf{R}^2$, it can be uniformized by a $\Gamma(y)$ relation, hence there is an $x \in \Delta(y)$ such that $x \in F'(|z|) = F'(\xi)$. By definition of F, $G(A_{\boldsymbol{d}}) \neq \xi$, a contradiction. □

Theorem 1 of course follows from 1.1 and 1.2.

REMARK. In 1.2 we actually proved more than was stated. The proof shows that any function H from the set of Δ-degrees into $\boldsymbol{\delta}$, for which $\forall \boldsymbol{d}$, $H(\boldsymbol{d}) \in A_{\boldsymbol{d}}$, is constant almost everywhere (in the sense of cones). This is also true for other types of degrees, for example Δ^1_m-degrees. For m even this proof works. For odd m a different proof is needed; Harrington–Kechris games [4] are used in place of the coding lemma and the uniformization theorem.

We now turn to a proof of Theorem 2. Ordinals less that $\boldsymbol{\delta}$ are coded as before. But countable ordinals are to be coded by reals in WO, that is by well orderings of ω. There is then a natural topology on codes for ordinals $\leqq \eta$ ($\eta < \omega_1$). Codes (reals) can be identified with elements of η^ω, and the topology on η^ω is the product topology, taking η discrete. This is the topology used to collapse η by forcing, where neighborhoods are determined by a finite sequence of ordinals less than η (conditions). Expressions such as a "comeager set of codes for η" refer to this topology.

For $\boldsymbol{d}$ a Δ-degree, let

$$B_{\boldsymbol{d}} = \{\xi < \boldsymbol{\delta} : \quad (\exists \text{ countable ordinals } \eta_1, \cdots, \eta_k)\ (\text{for a non-meager (in the product space) set of codes } w_1, \cdots, w_k \text{ for } \eta_1, \cdots, \eta_k)\ (\exists z \in \mathbf{R})\ (|z| = \xi \ \&\ z \text{ is } \Delta\text{-in-}(\boldsymbol{d}, w_1, \cdots, w_k))\}.$$

This essentially means that $\xi \in B_d$ if ξ has a code which is Δ in d and generic codes for countable ordinals.

LEMMA 2.1. *For all degrees* d, $\mathrm{card}(B_d) \leqq \aleph_1$.

PROOF. There are only $\aleph_1$ finite sequences $\eta_0, \cdots, \eta_k$ of countable ordinals. For each such sequence there are only countably many basic neighborhoods in the space of codes. And since AD implies that every set has the Baire property, non-meager is the same as comeager in a neighborhood. So it will suffice to show that for any fixed $\eta_1, \cdots, \eta_k$, and for any fixed neighborhood $N = N_1 \times \cdots \times N_k$ in the space of codes for $\eta_1, \cdots, \eta_k$, the following set is countable:

$$C = \{\xi < \delta : \quad \text{(for a comeager-in-}N\text{ set of codes } w_1, \cdots, w_k \text{ for } \eta_1, \cdots, \eta_k)\,(\exists z \in \mathbf{R})\,(|z| = \xi \;\&\; z \text{ is } \Delta\text{-in-}(d, w_1, \cdots, w_k))\}.$$

A well ordered union of meager sets is meager — this is a consequence of the Baire property. So there must be a fixed (i.e., independent of ξ) comeager-in-N set of codes $w_1, \cdots, w_k$ for $\eta_1, \cdots, \eta_k$ such that every element of C has a code Δ-in-$(d, w_1, \cdots, w_k)$. Hence C is countable. □

Let $\mathcal{U}_2$ be the ultrafilter on $P_{\aleph_2}(\delta)$ obtained from the map $d \mapsto B_d$ and the ultrafilter $\mathcal{V}$.

PROPOSITION 2.2. *$\mathcal{U}_2$ is a countably complete fine ultrafilter on $P_{\aleph_2}(\delta)$.*

Thus we know that $\mathcal{U}_2$ is $\aleph_1$-complete; but we must show that it is $\aleph_2$-complete. Call a filter $\mathcal{F}$ on $P_\kappa(\lambda)$ *very fine* if for any $T \in P_\kappa(\lambda)$, $\{S \in P_\kappa(\lambda) : T \subset S\} \in \mathcal{F}$.

PROPOSITION 2.3. *Let $\mathcal{U}$ be an ultrafilter on $P_\kappa(\lambda)$. If $\mathcal{U}$ is normal and very fine then $\mathcal{U}$ is κ-complete.*

LEMMA 2.4. *$\mathcal{U}_2$ is very fine.*

PROOF. Let $T \in P_{\aleph_2}(\delta)$. Let $f : \omega_1 \to T$ be a surjection. By the coding lemma, f is Γ, say $\Gamma(y)$ for $y \in \mathbf{R}$. Now for any $\eta < \omega_1$ and any code w for η, there is a code for $f(\eta)$ that is $\Delta(y, w)$. This uses the uniformization theorem for $\Gamma(y)$. Therefore if y is Δ-in-d, then by definition of B_d, for any $\eta < \omega_1$, $f(\eta) \in B_d$. So $T \subset B_d$ for a cone of degrees d. □

In light of 2.2–2.4, all that remains to be proved is that $\mathcal{U}_2$ is normal. The proof of this (2.6, below) is similar to the proof of 1.2; the idea of 1.2 will be combined with an idea used by Solovay in his orginal proof that $\aleph_2$ is measurable ([5], theorem 103b).

LEMMA 2.5. *$\forall y \in \mathbf{R}$, $\forall \eta < \omega_1$, there is a comeager set of codes w for η such that if d and d' are the Δ-degrees of y and $\langle y, w\rangle$, then $B_d = B_{d'}$.*

PROOF. Suppose not, and let y, η be a counterexample. Clearly $B_d \subset B_{d'}$, so for a non-meager set of codes w for η, there is a $\xi_w \in B_{d'(w)} \setminus B_d$, where $d'(w)$ is the degree of $\langle y, w\rangle$. Since a well ordered union of meager sets is meager, there is a fixed ξ such that $\xi = \xi_w$ for a non-meager set of w's. Using this fact again, applied to the definition of $B_{d'(w)}$, there must be a fixed $\eta_1, \cdots, \eta_k$ that works for a non-meager set of w's. That is, there exist countable ordinals $\eta_1, \cdots, \eta_k$ such that:

(For a non-meager set of codes w for η) (for a non-meager (in the k-dimensional product space) set of codes $w_1, \cdots, w_k$ for $\eta_1, \cdots, \eta_k$) $(\exists z \in \mathbf{R})(|z| = \xi \mathbin{\&} z$ is Δ-in-$(y, w, w_1, \cdots, w_k))$.

By the Kuratowski–Ulam theorem [13] (essentially the Product Lemma in forcing), this is equivalent to:

(For a non-meager (in the $k+1$-dimensional product space) set of codes $w, w_1, \cdots, w_k$ for $\eta, \eta_1, \cdots, \eta_k$) $(\exists z \in \mathbf{R})$ $(|z| = \xi \mathbin{\&} z$ is Δ-in-$(y, w, w_1, \cdots, w_k))$.

So by definition of B_d, $\xi \in B_d$, a contradiction. □

LEMMA 2.6. *$\mathcal{U}_2$ is normal.*

PROOF. Following the proof of 1.2, suppose not, let $G : P_{\aleph_2}(\boldsymbol{\delta}) \to \boldsymbol{\delta}$ be a choice function that is not constant $\mathcal{U}_2$-a.e., and let $F : \boldsymbol{\delta} \to \mathrm{Pow}(\mathbf{R})$ be the function:

$$\forall \xi < \boldsymbol{\delta},\ x \in F(\xi) \text{ iff (for all } \Delta\text{-degrees } \boldsymbol{d}) \text{ (if } x \text{ is } \Delta\text{-in-}\boldsymbol{d} \text{ then } G(B_d) \neq \xi).$$

Let $y \in \mathbf{R}$, let F' be a choice subfunction of F in $\Gamma(y)$, and let $\boldsymbol{d}$ be the Δ-degree of y. Let $\xi = G(B_d)$. Then there are $\eta_1, \cdots, \eta_k < \omega_1$ such that for a non-meager set of codes $w_1, \cdots, w_k$ for $\eta_1, \cdots, \eta_k$,

$$(*) \qquad (\exists z \in \mathbf{R})(|z| = \xi \mathbin{\&} z \text{ is } \Delta\text{-in-}(\boldsymbol{d}, w_1, \cdots, w_k)).$$

Using Lemma 2.5 k times, there exist codes $w_1, \cdots, w_k$ for $\eta_1, \cdots, \eta_k$ which satisfy $(*)$ and such that $B_d = B_{d'}$, where $\boldsymbol{d}'$ is the degree of $\langle y, w_1, \cdots, w_k\rangle$. Since there is a $z \in \Delta(y, w_1, \cdots, w_k)$ such that $|z| = \xi$, as in 1.2 there is an $x \in \Delta(y, w_1, \cdots, w_k)$ such that $x \in F'(|z|) = F'(\xi)$. By definition of F, $G(B_{d'}) \neq \xi$. But this contradicts the fact that $B_{d'} = B_d$ and $G(B_d) = \xi$. □

This completes the proof of Theorem 2.

The obvious question posed by these theorems is whether there are any more supercompact cardinals. The next candidate is $\aleph_{\omega+1}$, since the $\aleph_n$'s ($n \geqq 3$) are singular. There are many cardinals κ such that κ and κ^+ are both measurable, and this implies that κ is κ^+-supercompact. (This is due to Martin; see [2]). It is open whether there is any $\kappa > \aleph_2$ such that κ is κ^{++}-supercompact. The obstacle to generalizing the proof in this paper is the use of generic codes for countable ordinals. Kechris and Woodin [7] have recently discovered a type of "generic" code for uncountable ordinals. Unfortunately, the Kechris–Woodin codes do not have the property that a well ordered union of meager sets is meager, so the proof of Theorem 2 given here does not seem to generalize.

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